Pattern formation capacity of spatially extended systems

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We analyze a class of spatially extended systems which are capable of generating many complicated patterns. These systems are given by the Ginzburg-Landau equation coupled with a system of two linear equations and describe nonlinear media with localized defects. We find a connection between these systems and spin-glass systems. We show that the system is capable to produce many patterns and describe patterning algorithms.

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I. INTRODUCTION

In the last decade, great attention has been given to the problem of pattern formation and control. Complicated patterns can be observed in chemistry (e.g., Refs. [1–3]), physics (liquid crystals, magnetic thin films, Langmuir monolayers; see, e.g., Ref. [4] and references therein, Refs. [5,6]), biology (Refs. [7–11] among many others), or in polymers (e.g., Refs. [12–15]).

Since the general problem of pattern control is very difficult, it makes sense first to focus attention on simple fundamental models such as the Ginzburg-Landau and the complex Ginzburg-Landau equations, the Swift-Hohenberg equations, etc. Indeed, these equations are partial differential equation analogs of the normal forms for dynamical systems (Ref. [16]). They have been successfully used to analyze different patterns. Based on this approach, the pattern control theory has been developed in many papers (e.g., Refs. [17–23] for continuous models, Ref. [24] for coupled lattices).

The goal of this paper is to show that there is a connection between pattern formation problem and spin-glass models. During the last decades much attention was given to these spin-glass systems which are of great importance in neural networks and in other applications (Refs. [25–28]).

The results of this work can have applications both to patterning and neural networks.

Namely, we show that the long-range part of the expression for energy of some spatially extended systems coincides with the expression for energy of appropriate spin glasses. As such an extended system, we consider a model of Ginzburg-Landau type coupled with two spatially inhomogeneous linear equations.

The Ginzburg-Landau equation describes the time evolution of the order parameter u taking values close to 1 or -1. This model is used to study a nonlinear medium with well localized δ-like defects. It is a modification of the well known models describing the layered structure formation [4,12,14,15,29] in block copolymers, magnetic thin films, or smectic liquid crystals. As well it can be considered as a development of work [22] which showed how to control structure described by the complex Ginzburg-Landau equation by many organizing centers.

We find that for large times the energy of our system consists of two parts. The first is a short-range one (the Ginzburg-Landau energy). The second one is a long-range energy (cf. Refs. [4,30]). If the system parameters are adjusted in a special way, this second term can play a crucial role and define a pattern form. Then the system energy almost coincides with the energy of an associated spin-glass system.

For this model, we develop a method allowing us to control the nonlocal energy by choosing the coefficients appropriately. The long-ranged energy has the following form

$$E_{nonloc}=\sum_{ij}^{m} J_{ij}S_iS_j,$$

where S_i is the averaged value of the order parameter at ith localized defect: $S_i=\int_{\omega_i}u(x)dx$, and ω_i is a small ball containing the defect. We show that the matrix J depends linearly on the coefficients of the model and that, to obtain given J, it is enough to resolve an algebraic system of linear equations.

The most straightforward application of our results is to neural networks. Indeed, they yields that the Hopfield neural networks can be realized as parallel and locally controlled systems. Notice that in Hopfield neural networks [31] all the neurons are globally connected, and one cannot perform a

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local control. A price that one pays for the local control and the parallel architecture is an increase in the number of neurons.

On the other hand, our spatially extended systems possess a great capability to produce a large number of patterns.

We choose an inhomogeneous spatial perturbation as a sum of strongly localized (microscopical) defects. By these localized defects we can obtain different complicated patterns. The patterns consist of many domains where the order parameter is either 1 or -1. Under an appropriate choice of the system parameters, some set of microscopical impurities give rise to macroscopic domains, containing the impurities, where states of all the cells are identical. Thus, one can expect that, if the number of defects is large, the system is capable to produce many complicated patterns describing separation of the two phases. Recall that in Ref. [4] layered structures are studied. Here the patterns can be more complex. In particular, even using only some of the model parameters, one can obtain any convex polygons and unions of these polygons.

To conclude this introduction, notice that the Ginzburg-Landau contribution, even if it is relatively small, plays an important role. Namely, it smoothes and regularizes structures removing too complicated patterns with a great interface length. In consequence, the set of stable structures becomes more narrow and structures themselves are more regular. (The spin glasses with additional short-range interactions were studied, among others, in Ref. [32].)

The paper is organized as follows. In Sec. II we formulate our Ginzburg-Landau models. In Sec. III we describe some key physical and mathematical ideas. In Sec. IV the asymptotical solutions corresponding to narrow interface (''cell'') patterns are constructed. Here we also obtain an expression for the system energy. We prove that our system is almost gradientlike for large times. It implies the convergence of trajectories to local minima of the energy. Section V is focused on the problem of energy minimization. We study some particular cases, where minimizers can be obtained or where some qualitative properties of the minimizers can be described.

In Sec. VI, we find the connection between the pattern control problems and spin-glass systems. In this section we also describe control algorithms. Section VII concerns numerical simulations illustrating and complementing analytical approaches and results. At last, Sec. VIII contains final conclusions.

II. MODEL

We consider the following system of three equations:

$$u_t = \epsilon^2 \Delta u + u - u^3 + \lambda \gamma v, \tag{1}$$

$$\tau_1 v_t = d_1 \Delta v - a_1^2 v + b w, \tag{2}$$

$$\tau_2 w_t = d_2 \Delta w - a_2^2 w + \gamma u. \tag{3}$$

Here the coefficient γ is

$$\gamma(x,y) = \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i),$$

where $\mathbf{r} = (x,y) \in \Omega$ and $\mathbf{r}_i$ are distinct points of Ω and δ denotes an approximation of Dirac δ function. For example, we can set $\delta(\mathbf{r}) = 1/(\pi\rho^2)$ for $|\mathbf{r}| < \rho$, where ρ is a small radius. This choice of γ helps us to connect this model with spin-glass systems. [We will see that $S_i = u(\mathbf{r_i})$.] Moreover, we assume that $\epsilon > 0$ and $\lambda > 0$ are independent small parameters. We take the coefficient b depending on space coordinates, $b = b(x,y)$. It defines a feedback loop that is a key tool in the pattern control machinery.

Notice that simpler (and more fundamental) systems consisting of the Ginzburg-Landau equation coupled with one linear equation play the key role in many applications and were studied in Refs. [4,30,33].

Such systems can be also reduced to spin-glass models [if we insert δ-like perturbations similarly to Eq. (3)]. In this case, however, it is impossible to control completely the spin interaction matrix J. For the three component model this control exists due to the inhomogeneous coefficient $b = b(x,y)$ involved in the second equation (for details, see the Appendix). Roughly speaking we use the second linear equation to introduce the spins and the first one to control the matrix of the spin interaction. We thus think that system (1)–(3) is the simplest modification of the systems studied in Refs. [4,30] that can be reduced to spin-glass systems with a controllable spin interaction.

We investigate system (1)–(3) in a two dimensional domain Ω with a smooth boundary under the zero Neumann boundary conditions for the unknown functions u, v, and w:

$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0, \quad \frac{\partial w}{\partial n} = 0, \quad (x,y) \in \partial\Omega.$$

Also we discuss the one-dimensional (1D)-case, since it gives us simple test examples that can be examined analytically for small ϵ. Then $\Omega = [0,1]$ and $b = b(x)$, u, v, and w are functions of x,t.

The quantity u can be interpreted as an order parameter and Eq. (1) is the time dependent Ginzburg-Landau (TDGL) equation weakly perturbed by the term $\lambda \gamma v$. Equations (2) and (3) can describe, for example, an interaction of the order parameter with linear fields.

These fields play the important role creating a nonlocal long-range interaction between u states localized at different parts of our domain Ω. The characteristic radius R_{char} of this interaction has the order $a_1^{-1}\sqrt{d_1} + a_2^{-1}\sqrt{d_2}$. It is well known that a necessary condition for the pattern formation is that the v and w diffuse much faster than the order parameter u, i.e., $\epsilon^2 \ll d_i$.

III. KEY IDEAS

To investigate system (1)–(3) we use essentially the approach of the work [30].

To start our analysis, suppose first that $\lambda = 0$. Then Eq. (1) is the nonperturbed TDGL equation. Typically, it describes

interfaces' growth and their slow time evolution. The interfaces occur in a set I defined by the equation $u(x,y,0)=0$. Generically, I is a union of curves defined by the equations $R_l(x,y)=0$, $l=1,\ldots,m$. These lines separate domains Ω_k where the order parameter approaches 1 or -1 as the time increases.

The interface growth takes the time of order $|\ln\epsilon|$, and later a slow evolution of these lines starts. The interface curves $R_l(x,y,t)=0$ evolve, at each point (x,y), according to the mean curvature motion: $V_n=\epsilon^2\kappa$, where V_n is the normal interface velocity and κ is the interface curvature at this point (see Refs. [34,35] for details).

This dynamical process can be described by the energetic functional of the Ginzburg-Landau type associated with Eq. (1):

$$F_{GL}[u]=\frac{1}{2}\int_\Omega\left(\epsilon^2|\nabla u|^2+\frac{(1-u^2)^2}{2}\right)dxdy. \tag{4}$$

The energy F_{GL} is monotonically nonincreasing on solutions of Eq. (1) and thus all equilibrium stable configurations are given by local minima of energy (4). Nontrivial stable equilibria can exist only for nonconvex domains Ω. It is well known that, if ϵ is small and the pattern is an union of subdomains separated by interfaces, the Ginzburg-Landau energy is $\epsilon\,\mathrm{const}\,L$, where L is the complete interface length.

In the block copolymer theory, or if Eq. (1) is coupled with only a single additional linear equation through a term $\lambda\gamma v$ (where γ is a constant), the pattern, as ϵ tends to 0, stays the same: it is a partition of Ω to subdomains Ω_k, where $|u|=1$. These subdomains are separated by interfaces of width $O(\epsilon)$([4,30]). The equilibrium structures are given by local minima to the following nonlocal functional:

$$\begin{aligned}F[u]&=F_{GL}[u]+\frac{\lambda}{2}\int_\Omega\int_\Omega K(x,y|x',y')u(x,y)\\&\quad\times u(x',y')dxdydx'dy'\\&=F_{GL}[u]+\lambda F_{nonloc}[u],\end{aligned}$$

where K is a symmetric positive kernel. The nonlocal term describes the energy connected with long-range interactions in the system arising from coupling through v and u while F_{GL} is a gradient contribution connected with interface formation.

In contrast to F_{GL}, this nonlocal functional, in general, describes nontrivial stable structure even for convex domains. Simple examples of layered patterns observed in block copolymers are given in Refs. [12,14], a general theory is developed in Ref. [29].

In fact, the structure with a small number of interfaces is energetically non-advantageous since then F_{nonloc} becomes too great. On the other hand, if too many interfaces coexist, the gradient energy F_{GL} becomes very large. Finally, we have some optimal equilibrium interface number. This value is in an excellent accordance with the experimental data (Refs. [12,14]).

The key idea of our approach is the following. In our case, the equilibrium patterns are described by the following functional:

$$\begin{aligned}F[u]&=F_{GL}[u]+\frac{\lambda}{2}\sum_{i,j}J^b_{ij}u(x_j,y_j)u(x_i,y_i)\\&=F_{GL}[u]+\lambda F_{nonloc}[u],\end{aligned} \tag{5}$$

where the index b marks that the matrix J depends on the density $b(x,y)$. The main mathematical result of our analysis is as follows. In the 2D-case we can control completely the matrix J by a choice of this density. If λ is small, but much bigger than ϵ, we are able to describe analytically occurring complicated patterns. Indeed, we show that then the energy of the system has the same form as the energy of some spin-glass system. The kernel J defines the energy of this associated spin-glass system. All stable patterns of our system correspond to stable equilibria of this spin system. It is well known [36] that by changing J one can obtain many stable spin configurations.

However, as we will show, this control of J does not work for one-dimensional domains. In this case the control possibilities have been restricted: it is impossible to obtain any matrix J merely by changing $b(x)$.

IV. ASYMPTOTIC SOLUTIONS

Solutions describing the slow interface evolution are given by the following relations:

$$u=U_{int}+\tilde{u},$$

where U_{int} is the main contribution and $\tilde{u}$ is a small correction. The term U_{int} is a solution describing the interfaces. As we mentioned above, generically the interfaces are an union of curves. Near each curve the interface solution has the form $U_{int}=\tanh[\sqrt{2}\epsilon^{-1}n(x,y,t)]$ where n is the signed distance between the point (x,y) and the considered curve at time t, with $n>0$ if the point is in the domain where $u>0$ (see for details Refs. [34,35]).

For $\tau_i\ll 1$ and sufficiently large t the functions v and w can be approximated by the solutions V and W of the elliptic equations

$$d_1\Delta V-a_1^2V=-bW, \tag{6}$$

$$d_2\Delta W-a_2^2W=-\gamma U_{int}. \tag{7}$$

The solutions of Eq. (7) depend on time, since the right hand sides of this system are time dependent (through U_{int}). Physically, it is obvious that in the limit $\tau_i\to 0$ the dynamics of the fields v,w is captured by a slow interface dynamics.

Notice that the function V is of class $\mathcal{C}^2$.

The mathematical analysis can be done more rigorously and we will describe it in future publications.

Let us return now to Eq. (7). Denote

$$S_i = \int_{\omega_i} u(x,y)\,dxdy$$

the average of the order parameter at (x_i, y_i).

From Eq. (7) we obtain

$$W(x,y,t) = -\sum_{i=1}^{N} G_2(x,y|x_i,y_i)S_i, \tag{8}$$

where G_i, $i=1,2$ are the Green functions of the operators $d_i\Delta - a_i^2$.

Let us define the kernel K_b by the following relation:

$$K_b(x_1,y_1|x_2,y_2) = \int_\Omega G_1(x_1,y_1|x,y)b(x,y)G_2(x,y|x_2,y_2)dxdy.$$

Now by substituting Eq. (8) into Eq. (6) we have

$$V(x,y,t) = \sum_{i=1}^{N} S_i K_b(x,y|x_i,y_i).$$

This relation yields that asymptotically, for large times, the time evolution of the order parameter u is gradientlike. There is a functional of u (an "energy") that decreases along solutions of Eq. (1). Indeed, with our weakly perturbed TDGL Eq. (1) the following energy can be associated:

$$F = F_{GL} + \frac{\lambda}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} S_i S_j K_b^{sym}(x_i,y_i|x_j,y_j),$$

where the kernel K_b^{sym} is the symmetrization of K_b: $K_b^{sym} = \frac{1}{2}[K_b(x_1,y_1|x_2,y_2) + K_b(x_2,y_2|x_1,y_1)]$. Denoting $J_{ij}^b = K_b^{sym}(x_i,y_i|x_j,y_j)$ we obtain relation (5) announced above.

We will denote the nonlocal part of this energy by F_{nonloc}.

Thus one can expect that all solutions converge, for large times, to stationary patterns that minimize this energy (minimizers).

In the following section we consider these minimizers.

V. PATTERNS MINIMIZING ENERGY

Here we describe minimizers $u_{eq}(x,y)$ of the energy (5) giving equilibrium stable patterns. Each $u_{eq}(x,y)$ corresponds to a local minimum of F. To simplify our statement, first we consider a one-dimensional analog of functional (5). Below we extend our arguments to the 2D case.

Suppose, for the 1D case, that our domain Ω is the interval $[0,1]$. Then we have for small ϵ

$$F_{GL}[u] = \frac{1}{2}\int_0^1 \left(\epsilon^2 u_x^2 + \frac{(1-u^2)^2}{2}\right)dx \approx c\epsilon N_{int},$$

where N_{int} is the number of interfaces, c is a constant. Recall that we suppose $\epsilon \ll \lambda \ll 1$. To start our analysis, assume temporarily that $\epsilon = 0$. (It is clear, however, that this contribution may be important, even if ϵ is very small, and in the sequel we are going to take it into account). If $\epsilon = 0$, for the minimizers we have

$$u(x_i,y_i) = S_i^*, \tag{9}$$

where $S_i^* \in \{-1,1\}$ is a minimizer of the spin glass with the interaction J_{ij}^b. Clearly now for small ϵ the minimizers are formed by the minimal number of kinks (interfaces) such that Eq. (9) is satisfied. So, to find a minimizer, we operate in two steps: first we seek a spin pattern minimizing the associated spin-glass energy F_{spin}, second we find a kink structure with the minimal kink number consistent with (9).

This result is confirmed by numerical simulations (see Sec. VII). It holds under the conditions $\epsilon \ll \lambda \ll 1$. In fact, if $\lambda \ll \epsilon$ then the minimizer is trivial: it is $u \equiv \pm 1$. If, on the contrary, λ is order 1 or larger, then our problem cannot be asymptotically reduced to the spin-glass since $|S_i|$ are not close to 1: $|u| - 1 = O(\lambda)$.

Let us turn to the 2D case. Here we use the well known fact [34,35] that for small ϵ,

$$F_{GL}[u] \approx c\epsilon L_{int},$$

where L_{int} is the complete length of all interfaces separating zones where $u=1$ or $u=-1$.

First we find a spin distribution S_i^* that minimizes the spin-glass energy F_{spin}. Repeating the previous arguments, we must now seek a pattern with a minimal interface length consistent with the condition (9).

In some situations, this pattern can be described. If the S_i^* are all equal, which is the case if b takes only negative values, the optimal pattern consists of two zones: the convex envelope of the points $\mathbf{r}_i$ is occupied by one phase while the rest of the domain Ω is occupied by the opposite phase.

VI. SPIN-GLASS ENERGY CONTROL

We see that the energy of our system coincides with the energy of some spin-glass system, with a nonlocal spin interaction J_{kl}. These spin systems are well studied (Refs. [25–27]). Usually they have many stable spin configurations. Below, some ideas of our approach are a little bit similar to Ref. [36].

The problem on the nonlocal (spin) energy control reduces to the following: given a matrix J, to find a function $b(x,y)$ such that

$$2J_{kl} = \int_\Omega [G_1(x_k,y_k|x,y)G_2(x,y|x_l,y_l) + G_2(x_k,y_k|x,y)G_1(x,y|x_l,y_l)]b(x,y)dxdy. \tag{10}$$

This problem is resolvable even if the support b is localized inside some small open subdomain $B \subset \Omega$. A formal mathematical proof can be found in the Appendix.

Consider algorithms that allows us to find $b(x,y)$. Practically, to obtain a numerical method, we can consider

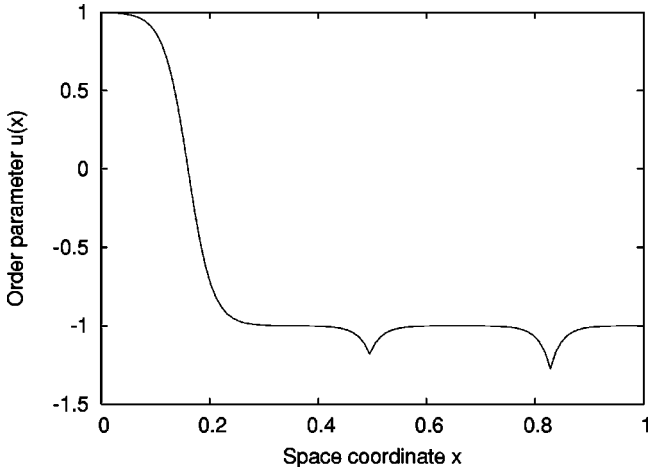

FIG. 1. Stationary solution u. The parameters are $\epsilon^2=0.001$, $\lambda=0.1$, $\tau_1=\tau_2=0.01$, $d_1=d_2=a_1=a_2=1$.

$b(x,y)$ as a sum of δ functions with weights w_k. If the number p of these δ peaks is large enough we can adjust the weights to satisfy Eq. (10). It is easy to see that it is enough to take at least $p=N^2$ points x^k,y^k chosen randomly and N^2 weights w_k. The choice of the weights can be performed by the least square method.

There is a second method that allows us to resolve the problem in an elementary way, however, this approach is valid under some conditions on the location of localized defects $\delta(\mathbf{r}-\mathbf{r}_i)$.

Namely, let us suppose that effective diffusion radii $R_i=a_i d_i^{-1/2}$, $i=1,2$ for v and w are much less than the distances $D_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$, $i\neq j$ between the localized defects. Let us assume that R_1 and R_2 have the same order R. We choose such b that $b(x,y)$ is zero for points (x,y) close enough to $\mathbf{r}_k$. Namely, we suppose that $b(x,y)$ is nonzero only if the minimal distance $D(\mathbf{r})$ between $\mathbf{r}=(x,y)$ and all localized defects $D(\mathbf{r})\gg R$ but $D(\mathbf{r})\ll D_{ij}$.

Then, in Eq. (10) we can use a rough asymptotic of the Green function

$$G_i(x,y|x_k,y_k)\approx\phi_i\exp(-|\mathbf{r}-\mathbf{r}_k|/R_i),$$

where the coefficient ϕ_i contains a nonexponential factor. Thus, the integral in Eq. (10) that defines J_{kl}, $k\neq l$ depends essentially only on the values $b(x,y)$ for points (x,y) on some interval of the right line that connects the points $\mathbf{r}_k$ and $\mathbf{r}_l$. If $k=l$ then $J_{kl}\approx 0$.

These values can be chosen independently since all intervals are disjoint. Thus system (10) splits then into $N(N-1)/2$ elementary independent equations.

It is difficult to investigate these algorithms numerically, because there are a number of choices of $b(x,y)$ and of the points (x_i,y_i). In the numerical simulations presented here, only simple choices of b are used.

VII. NUMERICAL SIMULATIONS

We first illustrate the 1D case. System (1)–(3) on the segment $[0,1]$ is discretized by finite differences, the diffusion is implicit and the reaction explicit.

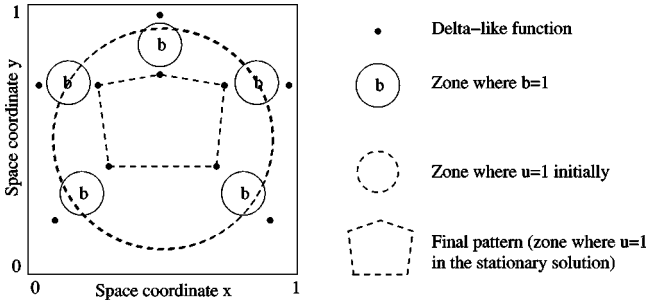

FIG. 2. Scheme of a choice of b, of the δ functions, and of the initial condition for u leading to the convex pentagon of Fig. 3.

For the case presented in Fig. 1, one considers three δ-like functions located at $x_1=1/6$, $x_2=1/2$, and $x_3=5/6$. The value of the control parameter is

$$b(x)=\begin{cases}1 & \text{if } x\in[0,1/3]\\ -1/2 & \text{if } x\in[1/3,1].\end{cases}$$

In this case, setting $d_1=d_2=a_1=a_2=1$ in Eqs. (2) and (3), one obtains $J_{12}>0$, $J_{13}>0$, and $J_{23}<0$. Hence the optimal configuration for the long-range energy satisfies

$$u(x_1)u(x_2)<0,\quad u(x_1)u(x_3)<0,\quad u(x_2)u(x_3)>0. \tag{11}$$

In Fig. 1 the stationary solution u of system (1)–(3) is plotted. It satisfies spin condition (11) and presents only one kink, hence confirming the arguments of Sec. V.

Note that to obtain the configuration of Fig. 1 the initial conditions have to be chosen properly. For the computation presented in Fig. 1, it is $u_0(x)=1$ for $x\in[0,1/3]$ and $u_0(x)=-1$ for $x\in[1/3,1]$. Hence the initial condition contains already one kink. Changing it to $u_0\equiv 1$ on $[0,1]$, one obtains a stationary solution without kink which does not satisfy condition (11).

Nonetheless the role of the control parameter b is crucial for the computation of u from Fig. 1. Changing it to

$$b\equiv -1/2 \quad \text{on } [0,1] \tag{12}$$

yields $J_{12}<0$, $J_{13}<0$, $J_{23}<0$, and replaces condition (11) by

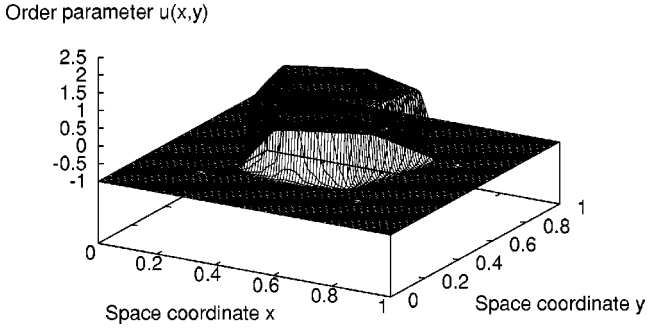

FIG. 3. Stationary solution u. The parameters are $\epsilon^2=3.75\times10^{-5}$, $\lambda=0.02$, $\tau_1=\tau_2=d_1=d_2=a_1=a_2=1$.

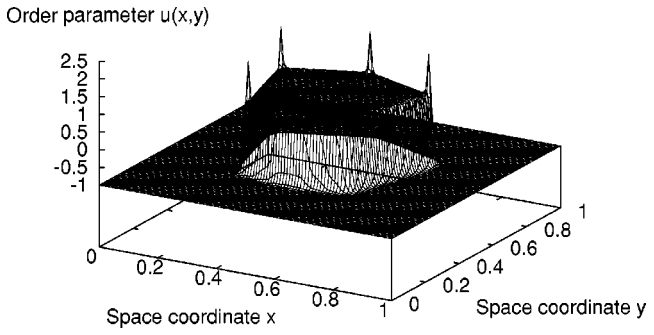

FIG. 4. Stationary solution u. The parameters are $\epsilon^2 = 3.75\times 10^{-5}$, $\lambda = 0.004$, $\tau_1 = \tau_2 = d_1 = d_2 = a_1 = a_2 = 1$.

$$u(x_1)u(x_2) > 0, \quad u(x_1)u(x_3) > 0, \quad u(x_2)u(x_3) > 0. \tag{13}$$

Setting all parameters as for Fig. 1 except b defined by Eq. (12) leads to a stationary solution without kink, satisfying condition (13) instead of condition (11).

We now turn to the 2D case. System (1)–(3) on the square domain $[0,1]\times[0,1]$ is discretized similarly as in the 1D case by using alternated directions.

For the computations presented in Figs. 3–5 the initial condition is $u_0(x,y) = 1$ for $(x,y) \in D((1/2,1/2),1/3)$ and $u_0(x,y) = 0$ otherwise [$D((1/2,1/2),1/3)$ denotes the disk of center (1/2,1/2) and radius 1/3]. In this case, if $\lambda = 0$ (u is then the solution of the non perturbed TDGL equation) the zone where $u(x,y) \neq -1$ vanishes and the stationary solution is $u \equiv -1$. When $\lambda \neq 0$, F_{nonloc} can change the final stationary solution.

Here we present two ways of achieving it. The first one illustrates the arguments of Sec. VI. We consider several ''packs'' constituted of two δ functions separated by a zone where b is positive. In this situation, if the effective diffusion radii are small enough, the long-range part of the energy is minimal when u has opposite sign at the two δ functions. This is the spin condition. Then, due to the Ginzburg-Landau part of the energy, the final pattern for u is a convex polygon whose edges are some of the δ functions (determined by the initial condition for u). In Fig. 2 is schematized such a choice of b, of the location of the δ functions, and of the initial condition for u. The final pattern is also schematized. In Fig. 3 is presented the resulting final pattern for u. This procedure allows to obtain any pattern constituted of an union of convex polygons.

This can also be obtained by a second procedure. In Figs. 4 and 5, one considers five δ-like functions constituting a convex pentagon included in the domain where $u \neq -1$ initially.

The control parameter b takes only negative values [in the computations presented $b(x,y) = -1$ for $(x,y) \in D((1/2,1/2),1/5)$ and $b(x,y) = 0$ otherwise] yielding that all the nondiagonal entries of the interaction matrix J be negative. The spin condition is then

$$u(x_i,y_i)u(x_j,y_j) > 0 \quad \text{for} \quad i,j = 1,2\ldots,5.$$

This prevents the zone where $u(x,y) \neq -1$ from vanishing.

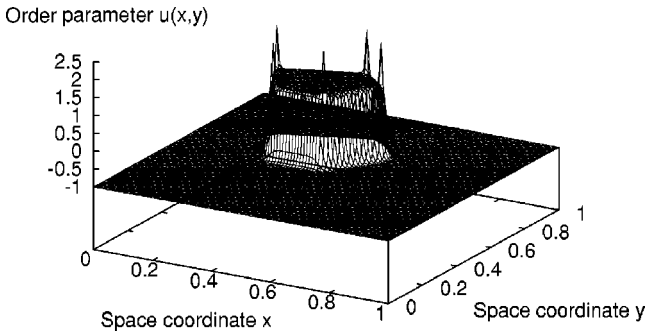

FIG. 5. Stationary solution u. Same parameters as Fig. 4.

In Figs. 4 and 5 the stationary solutions u are presented in this case. Note that the control parameter b in both of the graphs is the same. They differ only by the location of the δ-like functions. The solutions of Figs. 4 and 5 satisfy the above spin condition and also minimize the interface length, as expected by the arguments of Sec. V.

VIII. CONCLUSIONS

We have considered some spatially extended systems which are a natural generalization of models describing layered patterns [4]. Under an appropriate choice of their coefficients, it is possible to carry out a complete mathematical analysis of these models. It is based on a connection with spin-glass models that can be associated with these spatially extended systems.

It allows to show that the pattern capacity of some spatially extended systems is large. To be more precise, these systems can generate many patterns of complicated form, depending on initial data and parameters.

APPENDIX

Assuming the points $(x_k,y_k), k = 1,2,\ldots,p$ fixed, let us prove the solvability of Eq. (10) for any right hand sides J_{kl}. Denote $\mathcal{M}_p$ the space of all symmetric matrices $\mathbf{E}$ of size $p\times p$. Let us enable this space with the inner scalar product $(\mathbf{E},\mathbf{G}) = \Sigma_{ij} E_{ij} G_{ij}$. Consider the subset $\mathcal{M}^*$ of the matrices having the form

$$M_{kl} = \int_\Omega [G_1(x_k,y_k|x,y)G_2(x,y|x_l,y_l) + G_2(x_k,y_k|x,y)G_1(x,y|x_l,y_l)]b(x,y)dxdy$$

where b is a smooth function defined on B. To prove the solvability, we must show that $\mathcal{M}^* = \mathcal{M}_p$. Suppose the opposite. Then, as $\mathcal{M}^*$ is a closed subspace of $\mathcal{M}_p$, there exists a normal vector (i.e., a nonzero matrix) $\mathbf{E}$ orthogonal to the whole subspace $\mathcal{M}^*$. This means that

$$\sum_{kl} E_{kl} \int_B [G_1(x_k,y_k|x,y)G_2(x,y|x_l,y_l) + G_2(x_k,y_k|x,y)G_1(x,y|x_l,y_l)]b(x,y)dxdy = 0$$

for any b. Since b is arbitrary, this yields that

$$H(x,y)=\sum_{k,l} E_{kl}[G_1(x_k,y_k|x,y)G_2(x,y|x_l,y_l) + G_2(x_k,y_k|x,y)G_1(x,y|x_l,y_l)]\equiv 0 \quad \text{(A1)}$$

for any $(x,y)\in B$ and $(x,y)\neq(x_k,y_k)$. Notice that $H(x,y)$ is a real analytic function for all (x,y) outside of the boundary $\partial\Omega$ and such that $(x,y)\neq(x_k,y_k)$ since the Green functions G_1 and G_2 possess these properties (see, for example, Ref. [37]). It is well known that if a real analytic function is equal to zero in some open subdomain, then this function is equal to zero anywhere, where it is defined.

Thus, Eq. (A1) actually holds for any $(x,y)\neq(x_k,y_k)$, $k=1,2,\ldots,p$. Let us fix an index i. Consider (x,y) from a small neighborhood of the point (x_i,y_i). Near this point, the Green functions have the form

$$G_s(x,y|x_i,y_i)=c_s\ln[(x-x_i)^2+(y-y_i)^2]+\tilde{G}_s(x,y), \quad \text{(A2)}$$

for $s=1,2$, where $\tilde{G}_s$ are bounded and smooth as $(x,y)\rightarrow(x_i,y_i)$. Substituting relation (A2) into Eq. (A1) and investigating the behavior of the function H as $(x,y)\rightarrow(x_i,y_i)$, we see that the main singularity has the form $E_{ii}\{\ln[(x-x_i)^2+(y-y_i)^2]\}^2$ and the other singularities have smaller order. Thus, we conclude that $E_{ii}=0$ so that Eq. (A1) can be satisfied.

Now, again letting (x,y) tend to (x_i,y_i) and analyzing the logarithmic singularity of the left hand side of Eq. (A1) after the elimination of the term with $k=i,l=i$, we obtain

$$\tilde{H}_i(x,y)=\sum_{k,k\neq i} E_{ki}[c_1G_1(x_k,y_k|x,y)+c_2G_2(x_k,y_k|x,y)]\equiv 0$$

in a small neighborhood of (x_i,y_i). Repeating the previous arguments and using the analyticity of the Green functions outside of $\partial\Omega$ and the points (x_l,y_l), $l=1,2,\ldots,p$, we obtain that Eq. (A1) actually holds for all $(x,y)\neq(x_l,y_l)$, $l=1,2,\ldots,p$, inside Ω.

Since the index i can be chosen arbitrarily, it holds for any i. Now let us fix a new index j and let (x,y) tend to (x_j,y_j). The only singular term in Eq. (A1) has the coefficient $(c_1+c_2)E_{ij}$ where $c_1<0, c_2<0$.

Thus, $E_{ij}=0$ and we have proved that $\mathbf{E}=\mathbf{0}$.

This contradiction proves the solvability of system (10). Notice that in one-dimensional case this proof does not work and system (10) cannot be solved, at least for general J_{kl}. Actually, then the Green functions have the form $G_s(x|x_k)=C_s\exp(-\beta_s|x-x_k|)$ where $\beta_s=a_sd_s^{-1/2}$. Suppose that the domain $\Omega=[-1,1]$ and that the support of b is a small interval B centered at 0. Then, independently of the choice of b, all J_{kl} will have the same signs for any x_k and x_l outside of the support of b:

$$J_{kl}=C(b,\beta_1,\beta_2)\{\exp[-(\beta_1x_k+\beta_2x_l)] + \exp[-(\beta_2x_k+\beta_1x_l)]\}. \quad \text{(A3)}$$

We see from Eq. (A3) that the control of J_{kl} is impossible for the one-dimensional case.

[1] A.T. Winfree, in *Oscillations and Travelling Waves in Chemical Systems*, edited by R.J. Field and M. Burger (Wiley, New York, 1985), pp. 441–472.
[2] V. Castets, E. Dulos, J. Boissonade, and P. De Kepper, Phys. Rev. Lett. **64**, 2953 (1990).
[3] Q. Ouyang and H.L. Swinney, Nature (London) **352**, 610 (1991).
[4] T. Ohta and D. Jasnow, Phys. Rev. E **56**, 5648 (1997).
[5] A. Joets and R. Ribotta, Phys. Rev. Lett. **60**, 2164 (1988).
[6] H. Richter, A. Buka, and I. Rehberg, Phys. Rev. E **51**, 5886 (1995).
[7] H. Wager, Philos. Trans. R. Soc. London, Ser. B **201**, 333 (1911).
[8] S. Kondo and R. Asai, Nature (London) **376**, 765 (1995).
[9] T. Hofer and P.K. Maini, Nature (London) **380**, 678 (1996).
[10] A.M. Turing, Philos. Trans. R. Soc. London, Ser. B **237**, 37 (1952).
[11] J.D. Murray, *Mathematical Biology* (Springer-Verlag, Heidelberg, 1993).
[12] T. Ohta and K. Kawasaki, Macromolecules **19**, 2621 (1986).
[13] M. Bahiana and Y. Oono, Phys. Rev. A **41**, 6763 (1990).
[14] E. Aero, S. Vakulenko, and A. Vilesov, J. Phys. (France) **51**, 2205 (1990).
[15] S. Vakulenko, A. Vilesov, B. Stühn, and S. Frenkel, J. Chem. Phys. **106**, 3412 (1997).
[16] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[17] I. Aranson, H. Levine, and L. Tsimring, Phys. Rev. Lett. **72**, 2561 (1994).
[18] W. Lu, D. Yu, and R.G. Harrison, Phys. Rev. Lett. **76**, 3316 (1996).
[19] R. Martin, A.J. Scroggie, G.-L. Oppo, and W.J. Firth, Phys. Rev. Lett. **77**, 4007 (1996).
[20] R. Montagne and P. Colet, Phys. Rev. E **56**, 4017 (1997).
[21] M.E. Bleich, D. Hochheiser, J.V. Moloney, and J.E.S. Socolar, Phys. Rev. E **55**, 2119 (1997).
[22] S. Boccaletti, J. Bragard, and F.T. Arecchi, Phys. Rev. E **59**, 6574 (1999).
[23] J. Lega, Physica D **152-153**, 269 (2001).
[24] P. Parmananda, M. Hildebrand, and M. Eiswirth, Phys. Rev. E **56**, 239 (1997).
[25] M. Mezard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1988).
[26] V.S. Dotsenko, M.V. Feigel'man, and L.B. Ioffe, *Spin Glasses and Related Problems* of *Soviet Scientific Reviews* (Harwood, Chur, 1990), Vol. 15.
[27] D.L. Stein, *Spin Glasses and Biology* (World Scientific, Singapore, 1992).
[28] K.H. Fisher and J. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, 1991).

[29] C.B. Muratov, Phys. Rev. E **66**, 066108 (2002).
[30] D.M. Petrich and R.E. Goldstein, Phys. Rev. Lett. **72**, 1120 (1994).
[31] J.J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **79**, 2554 (1982).
[32] C.M. Newman and D.L. Stein, Phys. Rev. Lett. **76**, 515 (1996).
[33] L.M. Pismen, Phys. Rev. Lett. **86**, 548 (2001).
[34] J. Rubinstein, P. Sternberg, and J.B. Keller, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **49**, 116 (1989).
[35] J. Rubinstein, P. Sternberg, and J.B. Keller, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **49**, 1722 (1989).
[36] E. Gardner, J. Phys. A **21**, 257 (1988).
[37] M.A. Lavrent'ev and B.V. Shabat, *Methods of the Theory of Functions of Complex Variable* (Nauka, Moskva, 1973), in Russian.